Euclidean Spaces

- The most important geometric properties that make $\mathbb{R}^n$ so use are
  1. The existence of a scalar product
  2. A norm induced by the scalar product

- In this chapter, we will extend the notions of scalar product and norm to other vector spaces different to $\mathbb{R}^n$.

$\bar{x} \cdot \bar{y} = \langle \bar{x}, \bar{y} \rangle = x_1y_1 + x_2y_2 + x_3y_3$ usual <u>scalar product</u> in $\mathbb{R}^3$

Scalar Products

Let V be a vector space with scalars in $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$).

A scalar product on V is a function

$$\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{F}$$

$$(\bar{u}, \bar{v}) \mapsto \langle \bar{u}, \bar{v} \rangle$$

with the following properties:

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \bar{v} \neq \bar{0}$

1. $\langle \vec{v}, \vec{v} \rangle \geq 0 \; ; \forall \; \vec{v} \in V \; (\mathbb{R}), \quad \langle \bar{v}, \bar{v} \rangle \neq 0 \; (\mathbb{C})$
2. $\langle \vec{v}, \vec{v} \rangle = 0$ si y solo si $\vec{v} = \vec{0}$.
3. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle ; \; \forall \vec{u}, \vec{v}, \vec{w} \in V$
4. $\langle \alpha \vec{u}, \vec{v} \rangle = \alpha \langle \vec{u}, \vec{v} \rangle ; \; \forall \alpha \in \mathbb{F}, \; \forall \vec{u}, \vec{v} \in V.$
5. $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle} ; \; \forall \vec{u}, \vec{v} \in V$

$$\overline{\langle \bar{u}, \bar{v} \rangle} = \langle \bar{v}, \bar{u} \rangle \qquad \mathbb{R} \qquad \bar{a} = a$$

$$\langle \bar{u}, \bar{v} \rangle = \langle \bar{v}, \bar{u} \rangle$$

- If $\mathbb{F} = \mathbb{C}$, then $\forall \alpha \in \mathbb{C}$ and $\forall \vec{u}, \vec{v} \in V$

$$\langle \vec{u}, \alpha \vec{v} \rangle = \overline{\langle \alpha \vec{v}, \vec{u} \rangle} = \overline{\alpha \langle \vec{v}, \vec{u} \rangle} = \bar{\alpha} \langle \vec{u}, \vec{v} \rangle$$

▶ If $\mathbb{F}=\mathbb{R}$, then $\forall \alpha\in\mathbb{R}$ and $\forall \vec{u},\vec{v}\in V$

- $\langle \vec{u},\vec{v}\rangle = \overline{\langle \vec{v},\vec{u}\rangle} = \langle \vec{v},\vec{u}\rangle$
- $\langle \vec{u},\alpha\vec{v}\rangle = \alpha\langle \vec{u},\vec{v}\rangle$

$\boxed{\text{Corolary}}$

1. $\forall \vec{u}\in V,\ \ \langle \vec{0},\vec{u}\rangle = \langle \vec{u},\vec{0}\rangle = 0$
2. $\langle \sum_i a_i\vec{u}_i ,\ \sum_j b_j\vec{v}_i\rangle = \sum_{i,j} a_i\bar{b}_j\langle \vec{u}_i,\vec{v}_j\rangle$

$\forall a_i, b_j\in\mathbb{F}$ y $\forall \vec{u}_i,\vec{v}_j\in V.$

$\overline{ab}=\bar{a}\,\bar{b}$

▶ $\langle \bar{u},\alpha\bar{v}\rangle = \overline{\langle \alpha\bar{v},\bar{u}\rangle} = \overline{\alpha\langle \bar{v},\bar{u}\rangle}$

$= \bar{\alpha}\,\overline{\langle \bar{v},\bar{u}\rangle}$

$\langle \bar{u},\alpha\bar{v}\rangle = \bar{\alpha}\langle \bar{u},\bar{v}\rangle$

$\mathbb{F}=\mathbb{R}$: $\langle \bar{u},\alpha\bar{v}\rangle = \alpha\langle \bar{u},\bar{v}\rangle$

$\bar{a}=a$

$\overline{a+b}=\bar{a}+\bar{b}$

$$\langle \bar{u}, \bar{v}+\bar{w} \rangle = \overline{\langle \bar{v}+\bar{w}, \bar{u} \rangle} = \overline{\langle \bar{v}, \bar{u} \rangle + \langle \bar{w}, \bar{u} \rangle}$$

$$= \overline{\langle \bar{v}, \bar{u} \rangle} + \overline{\langle \bar{w}, \bar{u} \rangle}$$

$$\langle \bar{u}, \bar{v}+\bar{w} \rangle = \langle \bar{u}, \bar{v} \rangle + \langle \bar{u}, \bar{w} \rangle$$

$$\bar{v} - \bar{v} = \bar{0}$$

$$\langle \bar{0}, \bar{u} \rangle = \langle \bar{v}-\bar{v}, \bar{u} \rangle = \langle \bar{v}, \bar{u} \rangle - \langle \bar{v}, \bar{u} \rangle = 0$$

$$\langle \bar{u}, \bar{0} \rangle = \langle \bar{u}, \bar{v}-\bar{v} \rangle = \langle \bar{u}, \bar{v} \rangle - \langle \bar{u}, \bar{v} \rangle = 0$$

$$\langle a_1\bar{u}_1 + a_2\bar{u}_2 + a_3\bar{u}_3, b_1\bar{v}_1 + b_2\bar{v}_2 \rangle$$

$$= \langle a_1\bar{u}_1, b_1\bar{v}_1 + b_2\bar{v}_2 \rangle + \langle a_2\bar{u}_2, b_1\bar{v}_1 + b_2\bar{v}_2 \rangle$$

$$+ \langle a_3\bar{u}_3, b_1\bar{v}_1 + b_2\bar{v}_2 \rangle$$

$$= a_1 \langle \bar{u}_1, b_1\bar{v}_1 + b_2\bar{v}_2 \rangle + a_2 \langle \bar{u}_2, b_1\bar{v}_1 + b_2\bar{v}_2 \rangle$$

$$+ a_3 \langle \bar{u}_3, b_1\bar{v}_1 + b_2\bar{v}_2 \rangle$$

$$= a_1 \left(\bar{b}_1 \langle \bar{u}_1, \bar{v}_1 \rangle + \bar{b}_2 \langle \bar{u}_1, \bar{v}_2 \rangle \right)$$

$$+ a_2 \left(\bar{b}_1 \langle \bar{u}_2, \bar{v}_1 \rangle + \bar{b}_2 \langle \bar{u}_2, \bar{v}_2 \rangle \right)$$

$$+ a_3 \left(\bar{b}_1 \langle \bar{u}_3, \bar{v}_1 \rangle + \bar{b}_2 \langle \bar{u}_3, \bar{v}_2 \rangle \right)$$

$$= \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} \bar{b}_1 \langle \bar{u}_1, \bar{v}_1 \rangle + \bar{b}_2 \langle \bar{u}_1, \bar{v}_2 \rangle \\ \bar{b}_1 \langle \bar{u}_2, \bar{v}_1 \rangle + \bar{b}_2 \langle \bar{u}_2, \bar{v}_2 \rangle \\ \bar{b}_1 \langle \bar{u}_3, \bar{v}_1 \rangle + \bar{b}_2 \langle \bar{u}_3, \bar{v}_2 \rangle \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} \langle \bar{u}_1, \bar{v}_1 \rangle & \langle \bar{u}_1, \bar{v}_2 \rangle \\ \langle \bar{u}_2, \bar{v}_1 \rangle & \langle \bar{u}_2, \bar{v}_2 \rangle \\ \langle \bar{u}_3, \bar{v}_1 \rangle & \langle \bar{u}_3, \bar{v}_2 \rangle \end{pmatrix} \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \end{pmatrix}$$

$$\Uparrow$$

$$\langle a_1 \bar{u}_1 + a_2 \bar{u}_2 + a_3 \bar{u}_3 , b_1 \bar{v}_1 + b_2 \bar{v}_2 \rangle$$

$$\begin{array}{c|cc} & \bar{v}_1 & \bar{v}_2 \\ \hline \bar{u}_1 & \langle \bar{u}_1, \bar{v}_1 \rangle & \langle \bar{u}_1, \bar{v}_2 \rangle \\ \bar{u}_2 & \langle \bar{u}_2, \bar{v}_1 \rangle & \langle \bar{u}_2, \bar{v}_2 \rangle \\ \bar{u}_3 & \langle \bar{u}_3, \bar{v}_1 \rangle & \langle \bar{u}_3, \bar{v}_2 \rangle \end{array}$$

- It is possible to define many scalar products in a given vector space

- In a vector space V with scalar product $\langle \cdot , \cdot \rangle$, it is possible to write the product $\langle \vec{u}, \vec{v} \rangle$ $\forall \vec{u}, \vec{v} \in V$ in matrix form.

Fix a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$

$$\langle \vec{u}, \vec{v} \rangle = [\vec{u}]_{\mathcal{B}}^t \, G_{\mathcal{B}} \, [\vec{v}]_{\mathcal{B}}$$

where $G_{\mathcal{B}}$ is called the Gram matrix of $\langle \cdot , \cdot \rangle$ with respect to $\mathcal{B}$, given by

$$G_{\mathcal{B}} = \begin{pmatrix} \langle \vec{b}_1, \vec{b}_1 \rangle & \langle \vec{b}_1, \vec{b}_2 \rangle & \cdots & \langle \vec{b}_1, \vec{b}_n \rangle \\ \langle \vec{b}_2, \vec{b}_1 \rangle & \langle \vec{b}_2, \vec{b}_2 \rangle & \cdots & \langle \vec{b}_2, \vec{b}_n \rangle \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

$$\begin{vmatrix} \vdots & \vdots & & \vdots \\ \langle \vec{b}_n, \vec{b}_1 \rangle & \langle \vec{b}_n, \vec{b}_2 \rangle & \cdots & \langle \vec{b}_n, \vec{b}_n \rangle \end{vmatrix}$$

- Observe that $G_{\mathcal{B}} = \overline{G_{\mathcal{B}}}^{t}$.
- When V is a real vector space, $G_{\mathcal{B}} = G_{\mathcal{B}}^{t}$.

Example. Consider the usual scalar product on $\mathbb{R}^3$ defined by

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right\rangle = x_1y_1 + x_2y_2 + x_3y_3.$$

and the basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$$

Compute the Gram matrix of $\langle \cdot , \cdot \rangle$ with respect to $\mathcal{B}$.

$$\langle \bar{x}, \bar{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 = \underbrace{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}}_{= [\bar{x}]_{\mathcal{E}}} \overset{I_3}{\downarrow} \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{= [\bar{y}]_{\mathcal{E}}}$$

$$G_{\mathcal{E}} = I_3$$

$$G_{\mathcal{B}} = \begin{pmatrix} \langle \bar{b}_1, \bar{b}_1 \rangle & \langle \bar{b}_1, \bar{b}_2 \rangle & \langle \bar{b}_1, \bar{b}_3 \rangle \\ \langle \bar{b}_2, \bar{b}_1 \rangle & \langle \bar{b}_2, \bar{b}_2 \rangle & \langle \bar{b}_2, \bar{b}_3 \rangle \\ \langle \bar{b}_3, \bar{b}_1 \rangle & \langle \bar{b}_3, \bar{b}_2 \rangle & \langle \bar{b}_3, \bar{b}_3 \rangle \end{pmatrix}$$

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$G_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\bar{u} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \qquad \bar{v} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$[\bar{u}]_\varepsilon = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad [\bar{u}]_B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad [\bar{v}]_\varepsilon = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad [\bar{v}]_B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\langle \bar{u}, \bar{v} \rangle = [\bar{u}]_\varepsilon^t \underset{I_3}{\underset{\parallel}{G_\varepsilon}} [\bar{v}]_\varepsilon = \begin{pmatrix} 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 8$$

$$= [\bar{u}]_B^t G_B [\bar{v}]_B$$

$$= \underline{\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 8$$

Example. Consider the following scalar product defined on $\mathbb{P}_2$

$$\langle p,q\rangle = \int_0^1 p(x)\,q(x)\,dx\,, \quad \forall\, p(x), q(x) \in \mathbb{P}_2.$$

Compute the Gram matrix with respect to $\mathcal{E} = \{1, x, x^2\}$ and use it to compute

$$\int_0^1 (x-1)^2 \cdot 2(x-1)\,dx$$

$$\langle 1,1\rangle = \int_0^1 1\,dx = 1 \qquad \langle 1,x\rangle = \int_0^1 x\,dx = \frac{1}{2}$$

$$\langle 1,x^2\rangle = \int_0^1 x^2\,dx = \frac{1}{3} \qquad \langle x,x\rangle = \int_0^1 x^2\,dx = \frac{1}{3}$$

$$\langle x, x^2 \rangle = \int_0^1 x^3\,dx = \frac{1}{4} \qquad \langle x^2, x^2 \rangle = \int_0^1 x^4\,dx = \frac{1}{5}$$

$$\langle x, 1 \rangle = \langle 1, x \rangle$$

$$G_\varepsilon = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}$$

$$\int_0^1 p(x)q(x)\,dx = [p]_\varepsilon^t \, G_\varepsilon \, [q]_\varepsilon \qquad p, q \in \mathbb{P}_2$$

$$\int_0^1 (x-1)^2 \cdot 2(x-1)\,dx = [(x-1)^2]_\varepsilon^t \, G_\varepsilon \, [2(x-1)]_\varepsilon$$

$$[(x-1)^2]_\varepsilon = [1 - 2x + x^2]_\varepsilon = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$[2(x-1)]_\varepsilon = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$$

$$\int_0^1 (x-1)^2 \cdot 2(x-1)\,dx =$$

$$1/3 - 1/2 + 1/5$$

$$\frac{10-15+6}{30}$$

$$\begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$$

$$-1/30$$

$$= \begin{pmatrix} 1/3 & 1/12 & 1/30 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} = -2/3 + 1/6 = \frac{-4+1}{6} = -1/2$$

Scalar products and change of basis

- Any scalar product defined on a vector space is independent of the basis chosen to represent the vectors.

Let $\mathcal{B}$ and $\mathcal{C}$ be two bases of the vector space V, and let $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ be the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$.

$$[\vec{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\,[\vec{x}]_{\mathcal{B}}, \quad \forall \vec{x} \in V.$$

We will study the relationship between the Gram matrices with respect to two distinct bases.

For every $\vec{u}, \vec{v} \in V$

$$(AB)^t = B^t A^t$$

$$\begin{aligned}
\langle \vec{u}, \vec{v} \rangle &= [\vec{u}]_C^t \, G_C \, [\vec{v}]_C \\
&= \left(\underset{C \leftarrow B}{P} [\vec{u}]_B \right)^t G_C \left(\underset{C \leftarrow B}{P} [\vec{v}]_B \right) \\
&= [\vec{u}]_B^t \, \underset{C \leftarrow B}{P}^t \, G_C \, \underset{C \leftarrow B}{P} \, [\vec{v}]_B \\
&= [\vec{u}]_B^t \, G_B \, [\vec{v}]_B
\end{aligned}$$

Therefore: $G_B = \underset{C \leftarrow B}{P}^t \, G_C \, \underset{C \leftarrow B}{P}$

$T : \underset{B}{V} \to \underset{B}{V} \to M_T^{B} = \underset{B \leftarrow \tilde{B}}{P} \, M_T^{\tilde{B}} \, \underset{\tilde{B} \leftarrow B}{P}$

Example. Consider the usual scalar product in $\mathbb{R}^3$. Let

$$\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

two bases of $\mathbb{R}^3$. Verify the relation between $G_{\mathcal{B}}$ and $G_{\mathcal{E}} = I_3$

$$G_B = P^t_{\mathcal{E} \leftarrow B} \, G_{\mathcal{E}} \, P_{\mathcal{E} \leftarrow B} \qquad G_{\mathcal{E}} = P^t_{B \leftarrow \mathcal{E}} \, G_B \, P_{B \leftarrow \mathcal{E}}$$

$$P^{-1}_{\mathcal{E} \leftarrow B} \qquad P^{-1}_{\mathcal{E} \leftarrow B}$$

$$\underset{\mathcal{E}\leftarrow B}{P} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad G_{\mathcal{E}} = I_3$$

$$\underset{\mathcal{E}\leftarrow B}{P}^{t} \; G_{\mathcal{E}} \; \underset{\mathcal{E}\leftarrow B}{P}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \overset{\overset{G_B}{\parallel}}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}}$$

Example. Let U_{w_0} be the subspace of periodic functions with period $T_0 = 2\pi/w_0$ spanned by $\qquad j=\sqrt{-1}$

$$B = \{1, \cos w_0 t, \cos 2w_0 t, j\sin w_0 t, j\sin 2w_0 t\}.$$

That is, we are talking about the space of functions of the form

$$f(t) = a_0 + a_1 \cos w_0 t + a_2 \cos 2w_0 t + b_1 j \sin w_0 t + b_2 j \sin 2w_0 t.$$

Another basis for this space is

$$C = \{e^{-j2w_0 t}, e^{-jw_0 t}, 1, e^{jw_0 t}, e^{j2w_0 t}\}$$

$$e^{-j2w_0 t} = \cos(2w_0 t) - j\sin(2w_0 t)$$

These two bases are related by

$$\cos n\omega_0 t = \tfrac{1}{2} e^{jn\omega_0 t} + \tfrac{1}{2} e^{-jn\omega_0 t} \qquad n = 1, 2$$

$$j \sin n\omega_0 t = \frac{1}{2} e^{jn\omega_0 t} - \frac{1}{2} e^{-jn\omega_0 t} \qquad n = 1, 2$$

Consider the scalar product

$$\langle f(t), g(t) \rangle = \int_{T_0} f(t)\overline{g(t)}\, dt$$

(the scalars are complex numbers).

$T_0 = 2\pi$

$\omega_0 = 1$

$$\langle f(t), g(t) \rangle = \int_0^{2\pi} f(t)\overline{g(t)}\, dt$$

$$\langle 1, 1 \rangle = \int_{T_0} 1 \cdot 1\, dt = t \Big|_0^{T_0} = T_0 \qquad \langle 1, \cos \omega_0 t \rangle = \int_{T_0} \cos \omega_0 t\, dt = 0$$

The respective Gram matrices are

$$G_{\mathcal{B}} = \begin{pmatrix} T_0 & 0 & 0 & 0 & 0 \\ 0 & T_0/2 & 0 & 0 & 0 \\ 0 & 0 & T_0/2 & 0 & 0 \\ 0 & 0 & 0 & T_0/2 & 0 \\ 0 & 0 & 0 & 0 & T_0/2 \end{pmatrix}$$

$$G_C = \begin{pmatrix} T_0 & 0 & 0 & 0 & 0 \\ 0 & T_0 & 0 & 0 & 0 \\ 0 & 0 & T_0 & 0 & 0 \\ 0 & 0 & 0 & T_0 & 0 \\ 0 & 0 & 0 & 0 & T_0 \end{pmatrix} = T_0 I_5$$

The change of basis matrix $\underset{C\leftarrow B}{P}$ is

$$\underset{C\leftarrow B}{P} = \begin{array}{c} \begin{array}{ccccc} [1]_C & & & & \end{array} \\ \begin{pmatrix} 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 1/2 & 0 & -1/2 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \end{pmatrix} \end{array}$$

$$\cos \omega_0 t = \frac{1}{2} e^{-j\omega_0 t} + \frac{1}{2} e^{j\omega_0 t}.$$

It can be easily verified that

$$G_B = \underset{C\leftarrow B}{P^t}\, G_C \underset{C\leftarrow B}{P} = T_0 \underset{C\leftarrow B}{P^t} \underset{C\leftarrow B}{P}$$

Linear systems

Norm of a vector

$x \in \mathbb{R}$ $\quad |x|$

$\bar{x} \in \mathbb{R}^2$ $\quad \sqrt{x_1^2 + x_2^2}$

$\boxed{\text{Definition}}$

Let V be a vector space and $\langle \cdot, \cdot \rangle$ be a scalar product in V. The norm of a vector $\bar{u} \in V$ is defined by

induced by $\langle \cdot, \cdot \rangle$

$$\|\bar{u}\| = \sqrt{\langle \bar{u}, \bar{u} \rangle}.$$

$|z| = x + iy$

- Observe that the norm is well-defined since $\langle \bar{u}, \bar{u} \rangle \geq 0$ for all $\bar{u} \in V$.

- The norm depends on the scalar product.

Properties of the norm

1. $\|\vec{u}\| \geq 0.$ $\quad \sqrt{\underbrace{\langle \bar{u}, \bar{u} \rangle}_{\geq 0}}$
2. $\|\vec{u}\| = 0 \Leftrightarrow \vec{u} = \vec{0}$ $\quad |\alpha| > 1$
3. $\|\alpha \vec{u}\| = |\alpha| \, \|\vec{u}\|$ $\quad |\alpha| < 1$

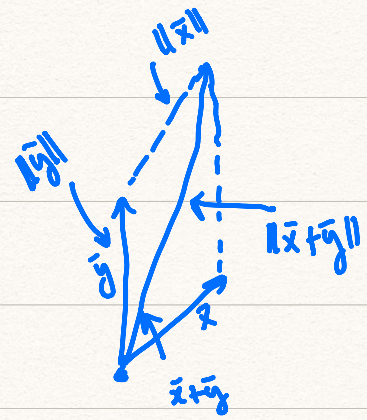
- A vector $\bar{u} \in V$ with $\|\bar{u}\| = 1$ is called a unit vector.
- For any vector $\bar{v} \neq \bar{0}$, a unit vector can be easily constructed

$$\frac{\pm \bar{v}}{\|\bar{v}\|} \qquad \left\langle \frac{\pm \bar{v}}{\|\bar{v}\|}, \frac{\pm \bar{v}}{\|\bar{v}\|} \right\rangle = \frac{(\pm 1)^2}{\cancel{\|\bar{v}\|^2}} \underbrace{\cancel{\langle \bar{v}, \bar{v} \rangle}}_{\|\bar{v}\|^2} = 1$$

Triangle inequality

For every $\bar{x}, \bar{y} \in V$:

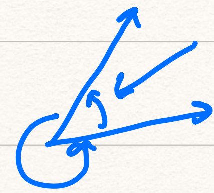
- $\|\bar{x} + \bar{y}\| \leq \|\bar{x}\| + \|\bar{y}\|.$



$\|\bar{x}\| = \sqrt{x_1^2 + x_2^2}$ $\qquad \|\bar{x} + \bar{y}\| = \sqrt{(x_1 + y_1)^2 + (x_2 + y_2)^2}$

$$\leq \sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2} = \|\bar{x}\| + \|\bar{y}\|$$

Angle between vectors



Schwarz Inequality

Let V be a vector space with scalar product $\langle \cdot , \cdot \rangle$. For every $\bar{x}, \bar{y} \in V$:

$$-\|\bar{x}\|\|\bar{y}\| \le \langle \bar{x}, \bar{y} \rangle \le \|\bar{x}\|\|\bar{y}\|$$

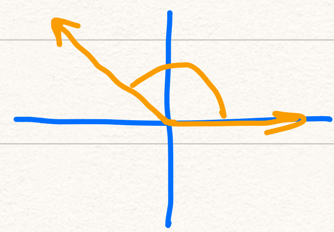
$$|\langle \bar{x}, \bar{y} \rangle| \le \|\bar{x}\| \cdot \|\bar{y}\| \Leftrightarrow \quad -1 \le \frac{\langle \bar{x}, \bar{y} \rangle}{\|\bar{x}\|\|\bar{y}\|} \le 1$$

$$\cos\theta = \frac{\langle \bar{x} \bar{y} \rangle}{\|\bar{x}\|\|\bar{y}\|} \rightarrow \theta = \arccos \frac{\langle \bar{x}, \bar{y} \rangle}{\|\bar{x}\|\|\bar{y}\|}$$

Definition

The angle between two vectors $\bar{x}$ and $\bar{y}$ is the unique number $0 \le \alpha \le \pi$ such that

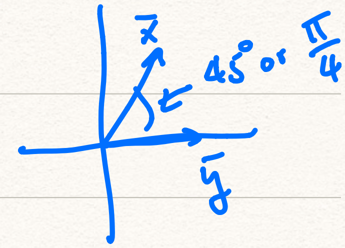
$$\cos\alpha = \frac{\langle \bar{x}, \bar{y} \rangle}{\|\bar{x}\| \cdot \|\bar{y}\|}$$



$$\langle \bar{x}, \bar{y} \rangle = \|\bar{x}\|\|\bar{y}\| \cos\alpha$$

Example Consider the two vectors

$$\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



- Consider the usual scalar product

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^t \vec{y}$$

Compute the angle between $\bar{x}$ and $\bar{y}$.

$$\cos\alpha = \frac{\langle \bar{x}, \bar{y} \rangle}{\|\bar{x}\| \|\bar{y}\|} = \frac{1}{\sqrt{2}} \qquad \alpha = \arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

- Consider the scalar product

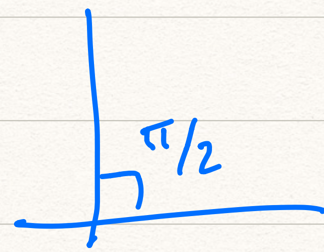
$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^t \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \vec{y}$$

$$\sqrt{\langle \bar{x}, \bar{x} \rangle} \quad \bar{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \rightarrow = 3x_1 y_1 + x_2 y_2$$

Compute the angle between $\bar{x}$ and $\bar{y}$.

$$\cos\theta = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2} \qquad \theta = \frac{\pi}{6} \; (30°)$$

▶ Two vectors $\bar{x}$ and $\bar{y}$ are orthogonal if $\langle \bar{x}, \bar{y} \rangle = 0$, equivalently, if the angle between $\bar{x}$ and $\bar{y}$ is $\pi/2$.



Before x-mass break

Orthogonal and orthonormal bases.

Definition

Let V be a vector space with scalar product $\langle \cdot, \cdot \rangle$ and let $\mathcal{B} = \{\bar{e}_1, \dots, \bar{e}_n\}$ be a basis of V.
$\mathcal{B}$ is an orthogonal basis if $\langle \bar{e}_i, \bar{e}_j \rangle = 0$ when $i \neq j$.
If additionally $\|\bar{e}_i\| = 1$ for all i, the $\mathcal{B}$ is an orthonormal basis.

- This definition can be extended to infinite dimensional vector spaces.

Example. Both

$$\mathcal{B} = \{\cos n\omega_0 t\}_{n=0}^{\infty} \cup \{\sin n\omega_0\}_{n=1}^{\infty}$$

and

$$C = \{e^{jn\omega_0 t}\}_{n=-\infty}^{\infty}$$

are orthogonal bases of the space of periodic functions with scalar product

$$\langle f, g \rangle = \int_T f(t)\,\overline{g(t)}\,dt$$

and period $T = \frac{2\pi}{\omega_0}$. $m \neq n$

$$\int_0^T e^{jn\omega_0 t}\,\overline{e^{jm\omega_0 t}}\,dt = \int_0^T e^{j(n-m)\omega_0 t}\,dt$$

$$= \frac{1}{j(n-m)\omega_0} e^{j(n-m)\omega_0 t}\Big|_0^T = \frac{\overbrace{e^{j(n-m)\omega_0 T}}^{=1} - 1}{j(n-m)\omega_0} = 0$$

$$T = \frac{2\pi}{\omega_0} \;\rightarrow\; \omega_0 = \frac{2\pi}{T}$$

$$j(n-m)\omega_0 T = j(n-m)\frac{2\pi}{T} \cdot T = j2\pi(m-n)$$

$$e^{j2\pi(n-m)} = \underbrace{\cos(2\pi(n-m))}_{1} + j\underbrace{\sin(2\pi(n-m))}_{0}$$

$$= 1 + j0$$

$$= 1$$

Example. The set of polynomials

$$\mathcal{B} = \left\{ 1, x, \tfrac{1}{2}(3x^2-1), \tfrac{1}{2}(5x^3-3x) \right\}$$

is an orthogonal basis of $\mathbb{P}_3$, with scalar product

$$\langle p,q \rangle = \int_{-1}^{1} p(x)\, q(x)\, dx.$$

$$\langle 1, x \rangle = \int_{-1}^{1} x\, dx = \frac{1}{2} x^2 \Big|_{-1}^{1} = \frac{1}{2}(1)^2 - \frac{1}{2}(-1)^2 = 0$$

$$\langle x, \tfrac{1}{2}(3x^2-1) \rangle = \int_{-1}^{1} \tfrac{1}{2}(3x^3 - x)\, dx$$

$$= \frac{1}{2}\left[\frac{3x^4}{4} - \frac{1}{2}x^2\right]_{-1}^{1} = \frac{1}{2}\left\{\frac{3}{4} - \frac{1}{2}\right\} - \frac{1}{2}\left\{\frac{3}{4} - \frac{1}{2}\right\} = 0$$

Example. The matrices $\qquad j = \sqrt{-1}$

$$\sigma_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \qquad \sigma_z = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \overline{\sigma_y^t}$$

form an orthogonal basis of the vector space of hermitian matrices $(A = \overline{A^t})$ of size 2×2 with scalar product defined by

$$\langle A, B \rangle = \mathrm{Trace}(\bar{A}^t B).$$

sum of the elements in the diagonal.

$$\langle \sigma_0, \sigma_x \rangle = \text{Trace}\left(\bar{\sigma}_0^t \sigma_x\right) = \text{Trace}\left(\sigma_0 \sigma_x\right) = 0$$

$$A = \bar{A}^t$$

$$\sigma_0 \sigma_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\|\sigma_0\| = \sqrt{\langle \sigma_0, \sigma_0 \rangle} = \sqrt{\text{Trace}\left(\sigma_0^2\right)} = 1$$

$$\sigma_0^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{1}{2} + \frac{1}{2} = 1$$

Orthonormal basis.

Quantum mechanics.

Exercise. Let V be a finite dimensional vector space with scalar product $\langle \cdot , \cdot \rangle$ and let $\mathcal{B} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ be a basis.

Show the following:

1. The basis $\mathcal{B}$ is orthogonal if and only if the Gram matrix w.r.t. B is diagonal.
2. The basis $\mathcal{B}$ is orthonormal if and only if the Gram matrix is the identity matrix.

$$G_B = \begin{pmatrix} \langle \bar{e}_1, \bar{e}_1 \rangle & \langle \bar{e}_1, \bar{e}_2 \rangle & \langle \bar{e}_1, \bar{e}_3 \rangle \cdots \\ \langle \bar{e}_2, \bar{e}_1 \rangle & \langle \bar{e}_2, \bar{e}_2 \rangle & \langle \bar{e}_2, \bar{e}_3 \rangle \cdots \\ \langle \bar{e}_3, \bar{e}_1 \rangle & \langle \bar{e}_3, \bar{e}_2 \rangle & \langle \bar{e}_3, \bar{e}_3 \rangle \\ & & & \ddots \end{pmatrix}$$

If B is orthogonal, $\langle \bar{e}_i, \bar{e}_j \rangle = 0 \quad i \neq j$

$$\langle \bar{e}_i, \bar{e}_i \rangle = \|\bar{e}_i\|^2 = 1$$

$\boxed{\text{Theorem}}$

Let V be an n-dimensional vector space. The change of basis matrix between two orthonormal bases B and C satisfies

$$\underset{C \leftarrow B}{P}^t = \underset{C \leftarrow B}{P}^{-1} = \underset{B \leftarrow C}{P}$$

If B and C are orthonormal then $G_B = G_C = I$

$$\underbrace{G_C}_{I} = \underset{B \leftarrow C}{P}^t \underbrace{G_B}_{I} \underset{B \leftarrow C}{P} \implies I = \underset{B \leftarrow C}{P}^t \underset{B \leftarrow C}{P}$$

$$\underset{B \leftarrow C}{P}^t = \underset{B \leftarrow C}{P}^{-1}$$

$$P P^t = I$$

$$P = \left(\bar{p}_1 \quad \bar{p}_2 \quad \cdots \quad \bar{p}_n \right)$$

$$P^t = \begin{pmatrix} \bar{p}_1^{\,t} \\ \bar{p}_2^{\,t} \\ \vdots \\ \bar{p}_n^{\,t} \end{pmatrix}$$

$$P^t P = \begin{pmatrix} \bar{p}_1^{\,t}\bar{p}_1 & \bar{p}_1^{\,t}\bar{p}_2 & \cdots & \bar{p}_1\bar{p}_n \\ \bar{p}_2^{\,t}\bar{p}_1 & \bar{p}_2^{\,t}\bar{p}_2 & \cdots & \bar{p}_2^{\,t}\bar{p}_n \\ \vdots & & & \\ \bar{p}_n^{\,t}\bar{p}_1 & \cdots & \cdots & \bar{p}_n^{\,t}\bar{p}_n \end{pmatrix} = I$$

$\langle \bar{x}, \bar{y} \rangle = \bar{x}^t \bar{y}$

I

$$\bar{p}_i^{\,t}\bar{p}_i = 1 \qquad \bar{p}_i^{\,t}\bar{p}_j = 0 \qquad i \neq j$$

$\{\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n\}$ are orthonormal with respect to usual scalar product in $\mathbb{R}^n$

Theorem

Let V be a vector space with scalar product $\langle \cdot , \cdot \rangle$, and $\mathcal{B} = \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_n\}$ is an orthogonal basis. Then, for every vector $\vec{x} \in V$

$$\vec{x} = \frac{\langle \vec{x}, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \vec{u}_1 + \frac{\langle \vec{x}, \vec{u}_2 \rangle}{\|\vec{u}_2\|^2} \vec{u}_2 + \cdots + \frac{\langle \vec{x}, \vec{u}_n \rangle}{\|\vec{u}_n\|^2} \vec{u}_n.$$

- The representation in the previous theorem is called a Fourier Series.

$$\bar{x} = c_1 \bar{u}_1 + c_2 \bar{u}_2 + \cdots + c_j \bar{u}_j + \cdots + c_n \bar{u}_n$$

$$\langle \bar{u}_i, \bar{u}_j \rangle = 0 \qquad i \neq j$$

$$\|\bar{u}_i\|^2 = \langle \bar{u}_i, \bar{u}_i \rangle \neq 0$$

$$\langle \bar{x}, \bar{u}_j \rangle = \langle c_1 \bar{u}_1 + c_2 \bar{u}_2 + \cdots + c_j \bar{u}_j + \cdots + c_n \bar{u}_n, \bar{u}_j \rangle$$
$$= c_1 \underbrace{\langle \bar{u}_1, \bar{u}_j \rangle}_{0} + c_2 \underbrace{\langle \bar{u}_2, \bar{u}_j \rangle}_{0} + \underbrace{\cdots}_{0} + \boxed{c_j \langle \bar{u}_j, \bar{u}_j \rangle}$$
$$+ \underbrace{\cdots}_{0} + c_n \underbrace{\langle \bar{u}_n, \bar{u}_j \rangle}_{0}$$

$$\langle \bar{x}, \bar{u}_j \rangle = c_j \|\bar{u}_j\|^2 \qquad c_j = \frac{\langle \bar{x}, \bar{u}_j \rangle}{\|\bar{u}_j\|^2} \quad j = 1, 2, \ldots, n$$

$$\|\bar{u}_j\|^2 = 1 \Rightarrow c_j = \langle \bar{x}, \bar{u}_j \rangle \quad j = 1, 2, \ldots, n.$$

Example. Consider the usual scalar product in $\mathbb{R}^3$, the orthogonal basis and the vector

$$B = \left\{ \underset{\bar{b}_1}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}, \underset{\bar{b}_2}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}, \underset{\bar{b}_3}{\begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix}} \right\}, \quad \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Find the Fourier series of $\bar{x}$ with respect to $\mathcal{B}$.

$$\bar{x} = \underbrace{\frac{\langle \bar{x}, \bar{b}_1 \rangle}{\|\bar{b}_1\|^2}}_{4/3} \bar{b}_1 + \underbrace{\frac{\langle \bar{x}, \bar{b}_2 \rangle}{\|\bar{b}_2\|^2}}_{-1/2} \bar{b}_2 + \underbrace{\frac{\langle \bar{x}, \bar{b}_3 \rangle}{\|\bar{b}_3\|^2}}_{1/3} \bar{b}_3$$

$$\langle \bar{x}, \bar{b}_1 \rangle = \bar{x}^t \bar{b}_1 = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 4$$

$$\|\bar{b}_1\|^2 = \langle \bar{b}_1, \bar{b}_1 \rangle = \bar{b}_1^t \bar{b}_1 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3$$

$$\langle \bar{x}, \bar{b}_2 \rangle = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -1$$

$$\|\bar{b}_2\|^2 = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 2$$

$$\langle \bar{x}, \bar{b}_3 \rangle = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix} = 1/2 + 1 - 1 = 1/2$$

$$\|\bar{b}_3\|^2 = \begin{pmatrix} 1/2 & 1/2 & -1 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix} = 1/4 + 1/4 + 1 = 3/2$$

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{4}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix}$$

Exercise. Show that a set of non-zero orthogonal vectors is linearly independent.

$B = \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_r\}$ orthogonal

if $r = \dim V$, B is a basis.

$$c_1\bar{u}_1 + c_2\bar{u}_2 + \cdots + c_n\bar{u}_n = \bar{0} \iff \underbrace{c_1 = c_2 = \cdots c_n = 0}$$

$$0 = \langle \bar{0}, \bar{u}_j \rangle = \langle c_1\bar{u}_1 + c_2\bar{u}_2 + \cdots + c_n\bar{u}_n, \bar{u}_j \rangle$$

$$0 = c_j \underbrace{\langle \bar{u}_j, \bar{u}_j \rangle}_{\neq 0} \Rightarrow c_j = 0$$

orthogonality $+ \ \#B = \dim V \Rightarrow B$ is a basis

$\Updownarrow$

lin. ind.

Gram-Schmidt orthogonalization method.

Let V be an n-dimensional vector space and let $\mathcal{B} = \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_n\}$ be a basis of V. Then $\{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\}$ is an orthogonal basis, where

$$\vec{e}_1 = \vec{u}_1$$

$$\vec{e}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{e}_1 \rangle}{\|\vec{e}_1\|^2} \vec{e}_1$$

$$\vdots$$

$$\vec{e}_i = \vec{u}_i - \lambda_{i,1} \vec{e}_1 - \dots - \lambda_{i,i-1} \vec{e}_{i-1}, \quad \lambda_{i,j} = \frac{\langle \vec{u}_i, \vec{e}_j \rangle}{\|\vec{e}_j\|^2}$$

$$\vdots$$

$$\vec{e}_n = \vec{u}_n - \underbrace{\frac{\langle \vec{u}_n, \vec{e}_1 \rangle}{\|\vec{e}_1\|^2} \vec{e}_1 - \dots - \frac{\langle \vec{u}_n, \vec{e}_{n-1} \rangle}{\|\vec{e}_{n-1}\|^2} \vec{e}_{n-1}}_{\text{Fourier series of } \bar{u}_n \text{ involving all the previous } \bar{e}_i\text{'s.}}$$

Example. Let $\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\vec{u}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

Consider the usual scalar product in $\mathbb{R}^3$ and the basis $\mathcal{B} = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$. Use Gram-Schmidt method to find an orthogonal and orthonormal basis of $\mathbb{R}^3$.

$\|\bar{e}_i\|^2 = 1$

$$\langle \bar{x}, \bar{y} \rangle = \bar{x}^t \bar{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$\bar{v} \neq \bar{0}$, $\dfrac{\bar{v}}{\|\bar{v}\|}$ is a unit vector.

$$\bar{e}_1 = \bar{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \|\bar{e}_1\|^2 = \langle \bar{e}_1, \bar{e}_1 \rangle = 1^2 + 1^2 + 1^2 = 3$$

$$\bar{e}_2 = \bar{u}_2 - \frac{\overbrace{\cancel{\langle \bar{u}_2, \bar{e}_1 \rangle}}^{0}}{\|\bar{e}_1\|^2}\bar{e}_1 = \bar{u}_2 \qquad \|\bar{e}_2\|^2 = 1^2 + (-1)^2 + 0^2 = 2$$

$$\langle \bar{u}_2, \bar{e}_1 \rangle = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$\bar{e}_3 = \bar{u}_3 - \frac{\overbrace{\cancel{\langle \bar{u}_3, \bar{e}_1 \rangle}}^{0}}{\|\bar{e}_1\|^2}\bar{e}_1 - \frac{\langle \bar{u}_3, \bar{e}_2 \rangle}{\|\bar{e}_2\|^2}\bar{e}_2 = \bar{u}_3 - \frac{1}{2}\bar{e}_2 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix}$$

$$\langle \bar{u}_3, \bar{e}_1 \rangle = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$\langle \bar{u}_3, \bar{e}_2 \rangle = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1$$

Orthogonal Basis: $\left\{ \bar{e}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \bar{e}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \bar{e}_3 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix} \right\}$

$$\|\bar{e}_3\|^2 = (1/2)^2 + (1/2)^2 + (-1)^2 = 1/4 + 1/4 + 1 = 3/2$$

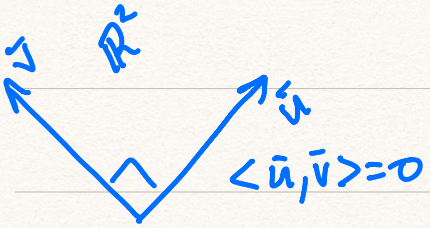
Orthonormal basis:

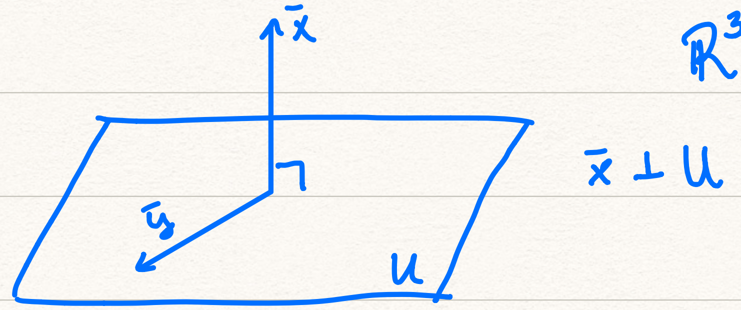
$$\left\{ \frac{\bar{e}_1}{\|\bar{e}_1\|}, \frac{\bar{e}_2}{\|\bar{e}_2\|}, \frac{\bar{e}_3}{\|\bar{e}_3\|} \right\} = \left\{ \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -\sqrt{2/3} \end{pmatrix} \right\}.$$

Orthogonal Complement

- Given a vector $\bar{x} \in V$ and a subspace $U \subseteq V$, we say that $\bar{x}$ is orthogonal to U, denoted by $\bar{x} \perp U$, if $\bar{x}$ is orthogonal to every vector in U. That is

$$\bar{x} \perp U \iff \langle \underset{\text{fixed}}{\bar{x}}, \underset{\text{any vector in } U}{\bar{y}} \rangle = 0, \ \forall \bar{y} \in U.$$





Theorem

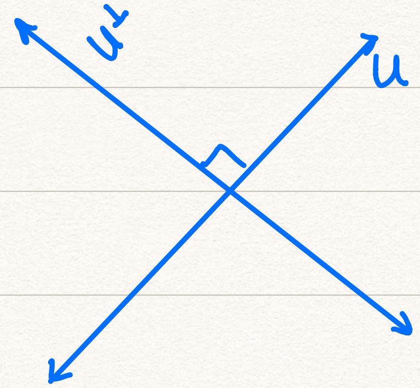
Let V be a vector space with scalar product $\langle \cdot , \cdot \rangle$, and let U be a subspace of V. Then, the set

$$U^{\perp} = \{ \vec{x} \in V, \ \vec{x} \perp U \}$$

called the orthogonal complement of U, is a subspace of V. Moreover, every vector $\bar{v} \in V$ can be written uniquely as $\bar{v} = \bar{x} + \bar{y}$ wher $\bar{x} \in U$ and $\bar{y} \in U^{\perp}$. In particular,

$$\dim U^{\perp} = \dim V - \dim U.$$

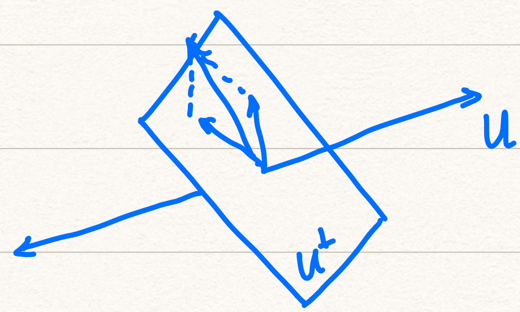
$\mathbb{R}^2$, usual $\langle \cdot, \cdot \rangle$



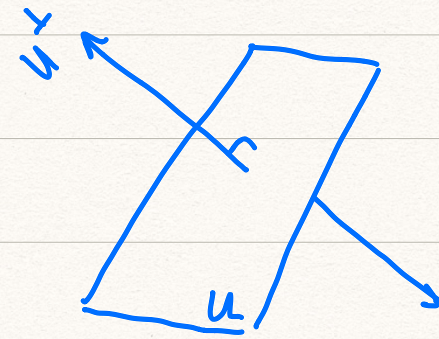
$$(\mathbb{R}^2)^\perp = \{\bar{0}\}$$

$$\langle \bar{0}, \bar{v} \rangle = 0$$

$\mathbb{R}^3$, usual $\langle \cdot, \cdot \rangle$



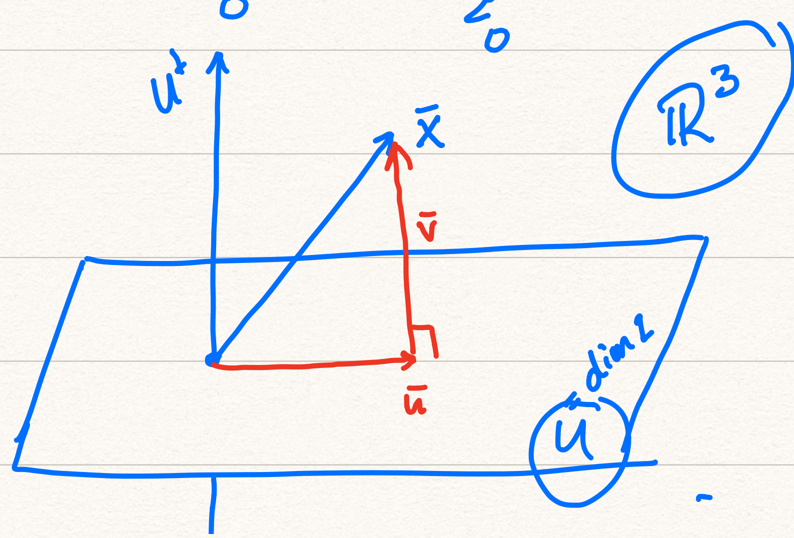
$$(U^\perp)^\perp = U$$



$U^\perp$ is a vector subspace

$\bar{x}, \bar{y} \in U^\perp \qquad \langle \bar{x}, \bar{u} \rangle = \langle \bar{y}, \bar{u} \rangle = 0 \quad \forall\, \bar{u} \in U.$

$$\langle \alpha\bar{x} + \beta\bar{y}, \bar{u} \rangle = \alpha \underbrace{\cancel{\langle \bar{x}, \bar{u} \rangle}}_{0} + \beta \underbrace{\cancel{\langle \bar{y}, \bar{u} \rangle}}_{0} = 0$$



$\downarrow$

Example. Consider the subspace of $\mathbb{R}^3$

$$U = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$ (usual scalar product)

Compute $U^\perp$.

$$\bar{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in U^\perp, \qquad \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \bar{x} \right\rangle = 0$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}}_{A}^{1\times 3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \qquad U^\perp = \mathrm{Nul}\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$$

$$\Updownarrow$$

$$\bar{x} \in U^\perp \iff \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \iff \bar{x} \in \mathrm{Nul}\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$$

$x + z = 0$

$y = \lambda$

$z = \mu$

$$U^{\perp} = \operatorname{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Example. In $\mathbb{R}^3$, consider the scalar product with Gram matrix with respect to the canonical basis

$$G = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

Consider the subspace

$$U = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Find $U^\perp$.

$G = I$ (previous example)

$$\langle \bar{x}, \bar{y} \rangle = \bar{x}^t G \bar{y}$$

$$\bar{x} \in U^\perp \Leftrightarrow \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \bar{x} \right\rangle = 0 \Leftrightarrow \bar{x} \in \text{Nul}\begin{pmatrix} 2 & 3 & 4 \end{pmatrix}$$

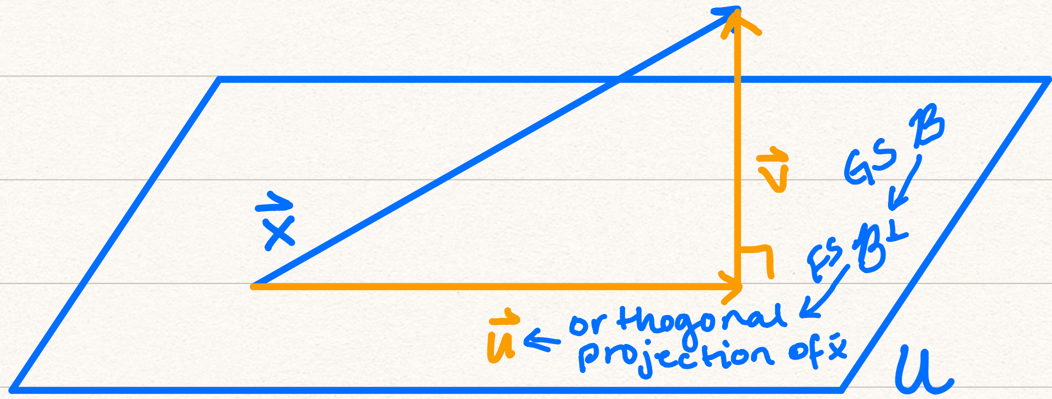
$$\Leftrightarrow$$

$$\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Orthogonal Projection

Every vector $\bar{x} \in V$ has a unique representation as $\bar{x} = \bar{u} + \bar{v}$ where $u \in U$ and $v \in U^{\perp}$.



▶ The vector $\bar{u} \in U$ is called the orthogonal projection of $\bar{x}$ over U and is denoted by $\hat{x}$.

Theorem

Let V be a vector space with scalar product $\langle \cdot, \cdot \rangle$, U is a subspace, and $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r\}$ an orthogonal basis of U.
Then, for every vector $\vec{x} \in V$,

$$\hat{x} = \frac{\langle \vec{x}, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \vec{u}_1 + \frac{\langle \vec{x}, \vec{u}_2 \rangle}{\|\vec{u}_2\|^2} \vec{u}_2 + \cdots + \frac{\langle \vec{x}, \vec{u}_r \rangle}{\|\vec{u}_r\|^2} \vec{u}_r.$$

Example. Let

$$\vec{u}_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix}$$

Let U be the subspace spanned by $\{\bar{u}_1, \bar{u}_2\}$. ~~Compute the projection operators over U and $U^\perp$ with the usual scalar product.~~

$$\bar{x} = \begin{pmatrix} 1 \\ -1 \\ 9 \end{pmatrix}$$

Compute the orthogonal projection of $\bar{x}$ onto U with respect to the usual scalar product.

I need an orthogonal basis for U. Are $\bar{u}_1$ and $\bar{u}_2$ orthogonal, or do I need to apply Gram-Schmidt?

$$\langle \bar{u}_1, \bar{u}_2 \rangle = \begin{pmatrix} 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = -3+2+1 = 0$$

Already orthogonal.

$$\hat{x} = \frac{\langle \bar{x}, \bar{u}_1 \rangle}{\|\bar{u}_1\|^2} \bar{u}_1 + \frac{\langle \bar{x}, \bar{u}_2 \rangle}{\|\bar{u}_2\|^2} \bar{u}_2$$

$$\hat{x} = \frac{\langle \bar{x}, \bar{u}_1 \rangle}{\underbrace{\|\bar{u}_1\|^2}_{1}} \bar{u}_1 + \frac{\langle \bar{x}, \bar{u}_2 \rangle}{\underbrace{\|\bar{u}_2\|^2}_{1}} \bar{u}_2 = \underbrace{\bar{u}_1 + \bar{u}_2}_{\text{component of } \bar{x} \text{ in } U}$$

$$\langle \bar{x}, \bar{u}_1 \rangle = \begin{pmatrix} 1 & -1 & 9 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = 3-1+9 = 11$$

$$\|\bar{u}_1\|^2 = \langle \bar{u}_1, \bar{u}_1 \rangle = \begin{pmatrix} 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = 11$$

$$\langle \bar{x}, \bar{u}_2 \rangle = \begin{pmatrix} 1 & -1 & 9 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = -1-2+9 = 6$$

$$\|\bar{u}_2\|^2 = \langle \bar{u}_2, \bar{u}_2 \rangle = \begin{pmatrix} -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 6$$

$$\bar{x} = \underbrace{\bar{u}_1 + \bar{u}_2}_{\hat{x}} + \underbrace{u_3}_{\in U^\perp}$$

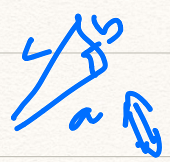
$$\bar{x} = \hat{x} + \bar{v}$$

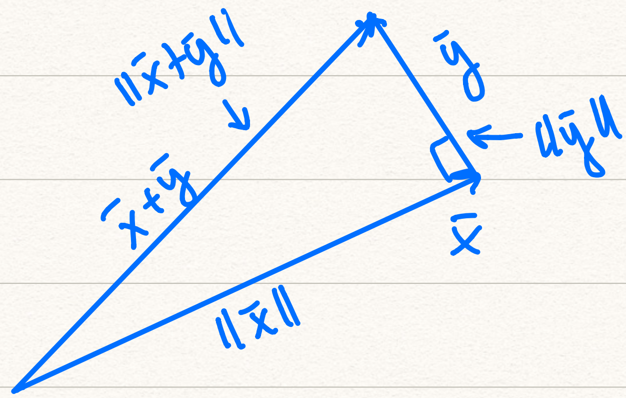
Theorem Pythagoras!!

Let V be a vector space with scalar product $\langle \cdot , \cdot \rangle$, and let $\bar{x}, \bar{y} \in V$, then $\bar{x} \perp \bar{y}$ if and only if

$$\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$$

$\mathbb{R}^2$, with the usual scalar product





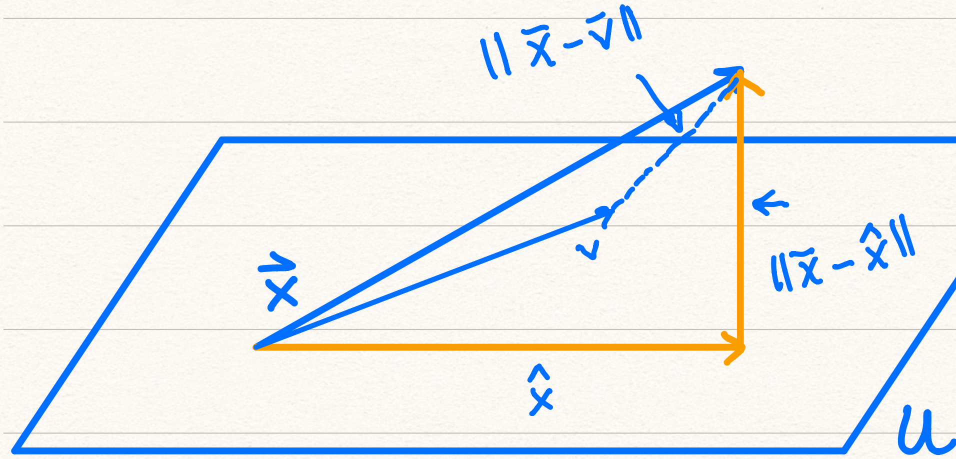
$$a^2+b^2=c^2$$

$$\bar{x} \perp \bar{y} \Leftrightarrow \|\bar{x}+\bar{y}\|^2 = \|\bar{x}\|^2 + \|\bar{y}\|^2$$

Theorem

Let V be a vector space with scalar product $\langle\cdot,\cdot\rangle$ and let U be a subspace of V. For every $\bar{x} \in V$, $\hat{x} \in U$ is the unique vector such that

$\forall \bar{v} \in U,\ \bar{v} \neq \hat{x},\ \|\bar{x} - \hat{x}\| < \|\bar{x} - \bar{v}\|$



$\|\bar{x} - \hat{x}\| < \|\bar{x} - \bar{v}\| \quad \forall \bar{v} \in U$

$\bar{v} \neq \hat{x}$